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PROPER GENERALIZED DECOMPOSITION (PGD) TO SOLVE MIXED CONVECTION PROBLEM

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Abstract. In this communication, the Proper Generalized Decomposition method (PGD), which is an a priori reduction model method, consisting in searching a solution of EDP in separated form, will be applied to solve non-isothermal Navier-Stokes equations. The performances of the PGD method will be compared to the standard resolution technique in term of CPU time as well as in term of accuracy.

1 INTRODUCTION

The numerical simulation of complex fluids flows leads to very large system that cannot be easily solved numerically. This situation is not convenient for optimization problems for which multiple solutions are usually required or for feed-back control problems for which real-time solutions are needed. Consequently, model reduction methods have been developed, as the Proper Orthogonal Decomposition (POD), the Central Voronoi Tesselations (CVT) or the A Priori Reduction method (APR) ...

In this communication we will focus on another model reduction method, the Proper Generalized Decomposition. It is an iterative method which consists in searching the solution u of an EDP in separating form:

$$u(x_1, ..., x_N) = \sum_{i=1}^{Q} \prod_{k=1}^{N} F_{ki}(x_k) \tag{1}$$

where the N variables x_i can be any scalar or vectorial variable, involving space, time or any parameter of the problem. Thus, if M degrees of freedom are used to discretize each variable, the total number of unknowns involved in the solution is $Q \times N \times M$ instead of the M^N degrees of freedom involved in traditional approaches. In most cases, where the filed is sufficiently regular, the number of terms Q in the finite sum is generally quite reduced (a few tens) and in all cases the approximation converges towards the associated solution. It must be emphasized here that the functions are not 'a priori' known. At each iteration they are adaptatively computed by introducing the approximation separated representation into the model and then by solving the resulting non-linear problem.

The PGD method has been applied by using a separation on space and time variables in order to solve solids mechanics problem [3], or stocahstic problems [2] (in this context, the PGD was initially called Generalized Spectral Decomposition). Method was extend to solve multidimensionals problems by Ammar et al. [1]. Finally, the method was aplied to solve isothermal 2D Navier-Stokes equations in the stationary and instationary cases with a separation only on the spaces variables [4, 5].

The aim of this communication is to show the ability of the PGD to treat non isothermal flows. After recall the general idea of the PGD, results for the case of the lid-driven cavity differentially heated will be presented.

2 "DESCRIPTION OF THE PGD

2.1 Preliminaries

For the sake of clarity and without losing its general scope, PGD will be examined in the case of a 2D space decomposition. The problem is expressed as follows :

$$\text{Find } U(x,y) \text{ as } \begin{cases} \mathcal{L}(U) = \mathcal{G} & \text{in } \Omega \\ +\text{Boundary Conditions} \end{cases} \tag{2}$$

where $\mathcal{L}$ is a linear[1] differential operator and $\mathcal{G}$ is the second member.

PGD consists in finding an approximation of the solution $U(x,y) \in \Omega = X \times Y \subset \mathbb{R}^2$ with $x \in X \subset \mathbb{R}$ and $y \in Y \subset \mathbb{R}$ as:

$$U(x,y) \approx U_m(x,y) = \sum_{i=1}^{m} \alpha^i F^i(x) G^i(y) \tag{3}$$

where $U_m(x,y)$ is the approximation of the solution of order m. At each iteration, the solution is enriched with an additional term $\alpha^{m+1} F^{m+1}(x) G^{m+1}(y)$. PGD should be decomposed in three steps. During the first step, "called the enrichment step", the F^{m+1} and G^{m+1} functions are obtained by solving a small size non-linear problem. Then, for the second step, called the "projection step", in order to improve the quality of the reconstruction, the $m+1$ α^i coefficients are determined by solving a linear system of size $(m+1)$. Finally, the "check convergence step" consists in the computing of the norm

[1] If the operator is not linear, it is necessary to linearize it.

of the residual in order to decide if the solution need more enrichment or not. In the following these three steps will be described in details.

2.2 Enrichment step

At the $m+1$ stage, the solution approximation of order m is supposed to be known. In this step we search to compute the functions $F^{m+1}(x)$ and $G^{m+1}(y)$. We search $U_m(x,y)$ as

$$U_m(x,y) = \sum_{i=1}^{m} \alpha^i F^i(x) G^i(y) + F^{m+1}(x) G^{m+1}(y) \tag{4}$$

By introducing equation (4) into problem (2) and by projecting onto each F^{m+1} and G^{m+1}, we obtain:

$$< \mathcal{L}(\sum_{i=1}^{m} \alpha^i F^i(x) G^i(y) + F^{m+1}(x) G^{m+1}(y)), F^{m+1} >_{L^2(X)} = < \mathcal{G}, F^{m+1} >_{L^2(X)} \tag{5}$$

$$< \mathcal{L}(\sum_{i=1}^{m} \alpha^i F^i(x) G^i(y) + F^{m+1}(x) G^{m+1}(y)), G^{m+1} >_{L^2(Y)} = < \mathcal{G}, G^{m+1} >_{L^2(Y)} \tag{6}$$

Solving this set of equations by a fixed point method, for example, gives the F^{m+1} and G^{m+1}.

2.3 Projection step

In order to increase the accuracy of the decomposition, the α^i coefficients are now searched in such a way that the residual is orthogonal to each of the $m+1$ products of the $F^i G^i$ functions. At this step, we search $U_{m+1}(x,y)$ as,

$$U_{m+1}(x,y) = \sum_{i=1}^{m+1} \alpha^i F^i(x) G^i(y) \tag{7}$$

By introducing solution (7) into (2) and by projecting it according each $F^i G^i$, we obtained the following system of equations:

$$< \mathcal{L}(\sum_{i=1}^{m+1} \alpha^i F^i(x) G^i(y)), F^k G^k >_{L^2(\Omega)} = < \mathcal{G}, F^k G^k >_{L^2(\Omega)} \qquad \text{for } 1 \leqslant k \leqslant m+1 \tag{8}$$

The resolution of this linear system gives the α^i coefficients .

2.4 Check convergence step

At this step the residual is computed in the following way :

$$Res^{m+1} = \mathcal{L}(\sum_{i=1}^{m+1} \alpha^i F^i(x) G^i(y)) - \mathcal{G} \tag{9}$$

If the L^2 norm of this residual is lower than a coefficient ϵ set by the user, the PGD algorithm was converged.

3 GOVERNING EQUATIONS OF NON ISOTHERMAL FLOW

• Let us a domain Ω with boundary Γ. Flow in the domain is considered as Newtonian, incompressible and constant properties. Boussinesq approximation is performed. Equations of the problem could be write in dimensionless form :

$$\begin{cases} \nabla.\mathbf{u} = 0 \\ \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u} = \dfrac{1}{Re}\Delta \mathbf{u} - \nabla p + Ri\theta \overrightarrow{y} \\ \dfrac{\partial \theta}{\partial t} + (\mathbf{u}.\nabla)\theta = \dfrac{1}{RePr}\Delta\theta \end{cases} \tag{10}$$

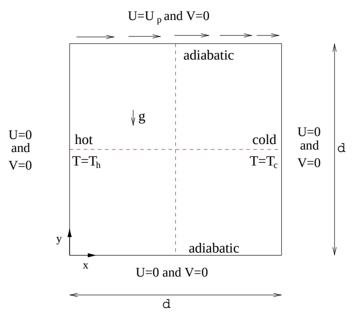
where $\mathbf{u} = (u_1, u_2)$ is the velocity field, θ the temperature and p the pressure. Re is the Reynolds number, Pr the Prandtl number and Ri is the Richardson number which characterizes the mixed convection flow.

These equations are solved by using a time splitting scheme, the so-called Van Kahn algorithm and a finite volume method.

• Solve this problem by PGD consists in searching the unknowns at each time step n in the following form :

$$u_i^n = \sum_{l=1}^{N_{U_i}} \alpha_{U_i}^l F_{U_i}^l(x) G_{U_i}^l(y) \quad p^n = \sum_{l=1}^{N_P} \alpha_P^l F_P^l(x) G_P^l(y) \quad \theta^n = \sum_{l=1}^{N_\Theta} \alpha_\Theta^l F_\Theta^l(x) G_\Theta^l(y) \tag{11}$$

4 APPLICATION OF THE PGD TO THE LID DRIVEN CAVITY



The definition sketch of the problem and the boundary counditions are shown in left figure. It is a square cavity with an incompressible fluid. Vertical walls have different temperature and horizontals walls are adiabatic. The top wall is moving right, and the the two velocity components vanish on the three others walls.

Simulations are made with $\delta t = 10^{-3}$ and for three Richardson number ($Ri = 0.1$, $Ri = 1$ and $Ri = 10$). Results of PGD will be compared to results of a standard solver (Bi-Conjugate gradient solver) in order to compare the accuarcy and CPU time between both methods.

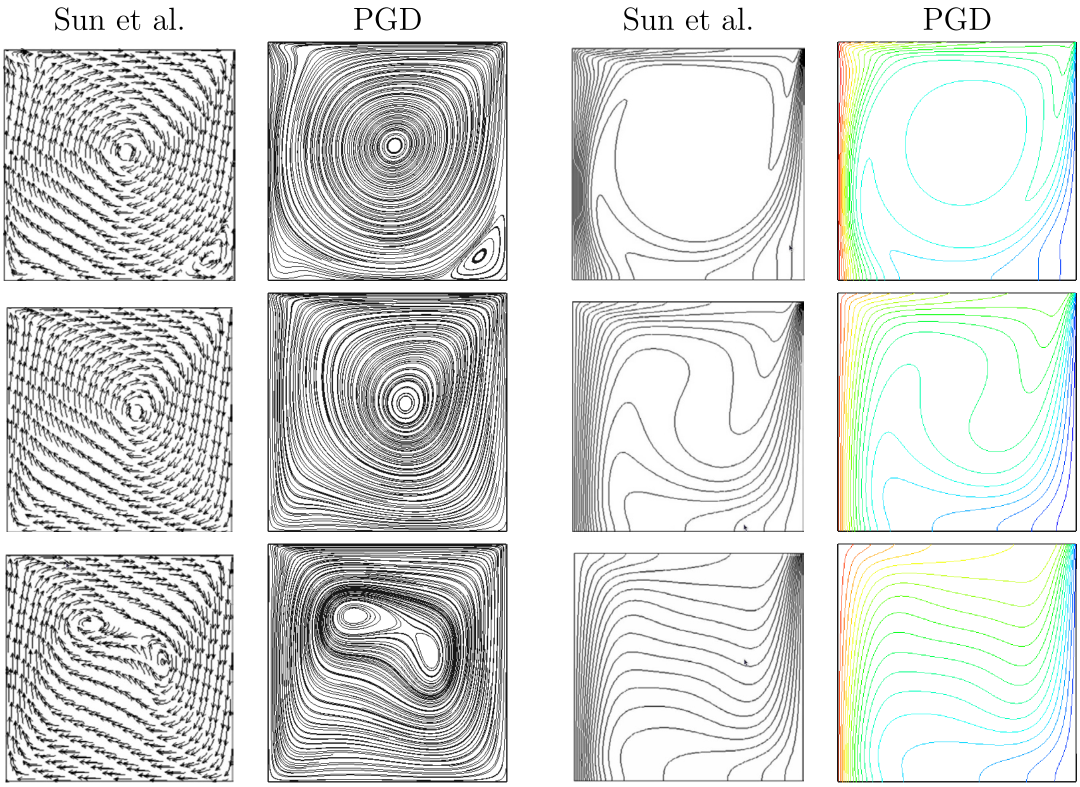


Figure 1: Comparison of the Streamline of veloity obtained by PGD with these from literature (left) and isovalue of temperature field obtained by PGD and from literature(right) for $Ri = 0.1$ (top), $Ri = 1$ (middle), $Ri = 10$ (bottom)

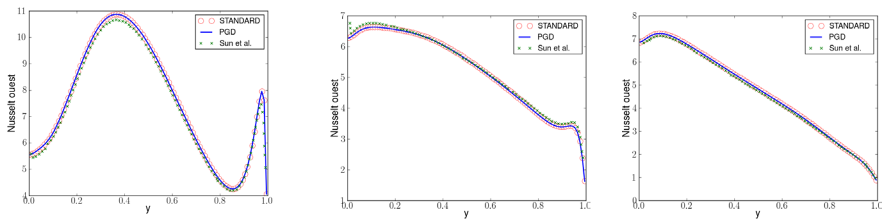


Figure 2: Local nusselt number at west for $Ri = 0.1$ (left), $Ri = 1$ (middle), $Ri = 10$ (right)

Figure 1 shows that the streamlines and isovalue of temperature obtained by PGD are similar to these obtained by Sun et al. [6]. Moreover, figure 2 shows that local Nusselt numbers which is defined by $Nu_w = -(\partial\theta/\partial X)_w/(\theta_h - \theta_c)$, computed from PGD and standard solver and from [6] are similar. About the computational duration, figure 3 shows that from approximatively 150 nodes in each direction, PGD becomes faster than standards solver. For a mesh size of 500×500, the CPU time was three times lower with the PGD solver for $R1 = 0.1$ and $Ri = 1$ and six times lower for $Ri = 10$.

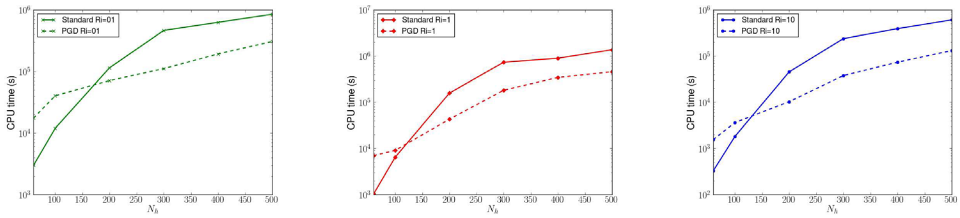


Figure 3: Computational duration in function of the number of nodes in each direction for $Ri = 0.1$ (left), $Ri = 1$ (middle), $Ri = 10$ (right)

4.1 Conclusion

We have observed that the PGD is able to solve this problem accurately with a CPU time saving in relation to the standard solver. This work is a first attempt to use PGD method to solve mixed convection problems. Some improvements are worth further developments. The extension to the 3D case is required to benefit more from the CPU time reduction. Furthermore, being able to consider time as a new variable of the tensorial decomposition could be an original alternative to the time integration scheme. However, the difficulty raised here will be related to the convergence rate of a 4D variables problem.

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